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ON THE INVARIANT CRITERIA FOR THE REALITY OF THE ROOTS OF THE QUINTIC.

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[With an Historical Introduction by Prof. James McMahon, Ithaca, N. Y.]

I. Historical Introduction.

The Sturmian functions (dating from 1835) still furnish the best criteria for examining the character of the roots of a numerical equation, on account of their low degree in the coefficients.

As generalized criteria, however, they have the disadvantage of not being expressed in terms of invariants.

Prior to 1854 the criteria that were invariantive in form, and known to be necessary conditions for the reality of the roots of the quintic, had the defect of not being sufficient conditions.

Thus the problem was to obtain criteria, both necessary and sufficient for reality, expressed in terms of the fundamental invariants.

The possibility of doing this was first shown in Hermite's classical paper on the quintic (Cambridge and Dublin Mathematical Journal, 1854, Vol. IX, p. 172), where it is proven that every binary quantic of odd degree is reducible by a real linear transformation to a *forme-type* in which the new variables are linear covariants, and the coefficients are invariants; for then Sturm's theorem applied to the *forme-type* gives criteria in the invariant form.

Ten years later appeared the "Trilogy" (Phil. Trans., 1864, p. 579), in which Sylvester, while acknowledging his debt to Hermite, derives simpler criteria by means of the new conceptions of "facultative points" and the "amphigenous surface." A brief sketch of this method is given in Salmon; but it is very desirable to consult the original article, as well as Cayley's extension to facultative points in m-space, in the Eighth Memoir on Quantics (Phil. Trans., 1867, p. 529).

In these articles the three fundamental invariants are: J, of order 4 and weight 10; K, of order 8 and weight 20; L, of order 12 and weight 30.

In terms of these may be expressed the discriminant $D = J^2 - 2^7K$, of order 8 and weight 20; and the criterion-invariant $\Lambda = 2^{11}L - J^3$, of order 12 and weight 30.

Sylvester's results are given on p. 643 of the "Trilogy," and may be tabulated as follows:

D	J	$-\Lambda$	Roots of quintic.
$-$			two imaginary
$+$	$-$	$-$	none imaginary
$+$	not both negative		four imaginary
0	$-$	$-$	none imaginary two equal
0	not both negative		two imaginary two equal
0	$-$	0	none imaginary two pairs equal
0	$+$	0	four imaginary two pairs equal
0	0	0	three roots equal.

He further shows geometrically that in these criteria Λ may be replaced by the invariant of the same order $\Lambda + \mu JD$, where μ may have any value between $+1$ and -2.

Hermite, in his next series of papers on the quintic (*Comptes Rendus*, 1866), expresses his admiration for the method that led to a result so important and so novel in Algebra, viz. a criterion involving a variable parameter within certain limits; he calls it "one of the most beautiful discoveries of the learned English geometer," but points out certain advantages in his own algebraic method, although not leading to criteria so simple in form.

The papers mentioned probably contain the most important additions to the present problem; indeed, Sylvester's Criteria being theoretically perfect, further contributions would be likely to take the form of algebraic proofs of his results.

The readers of the masterly discussions of Sylvester and Cayley will be interested in the following derivation of the Criteria by methods more algebraic, yet somewhat shorter; and it is well to observe here that the suggestion of the form of the criterion-invariant Λ is not necessarily taken from Sylvester's result; for Λ previously presents itself as the invariant whose vanishing along with D is the condition for an additional pair of equal roots. (See Salmon, Art. 228.)

The case just mentioned, and the others indicated above, in which the discriminant is zero or negative, present no difficulty; hence Dr. Harris treats only of the two cases in which the discriminant is positive.

He also applies his method to Sylvester's more general criterion involving a parameter μ, and shows how the limits between which μ may vary may be extended for particular quintics.

Hermite's result may also be explained and verified by the same method.

II. SYLVESTER'S SIMPLE INVARIANT CRITERIA.

In deriving the complete criteria for the roots of the Quintic, the only cases that require special investigation, as indicated in the historical note, are those in which the discriminant is positive, i. e. in which the number of imaginary roots is four or zero, and of real roots one or five. We shall, then, in what follows assume the discriminant to be positive; and we begin with an algebraic proof of Sylvester's simple invariant criteria for distinguishing these two cases.

1. In passing from the case of five real roots to that in which four of them shall have become imaginary, it is clear that we must pass through the case of two pairs of equal roots (cf. Art. 238).* This is easily understood by supposing the roots of a quintic to be the x-co-ordinates of the points of intersection of a straight line and a parabola of the fifth order. Suppose that certain coefficients involved in the equation of either, or in both equations, to undergo continuous variations. In passing from a line cutting the parabola in five real points to a line cutting it in but one real point, we must pass through a double tangent; in other words, through a quintic having two pairs of equal roots. We cannot lose one pair at a time, because the discriminant would then be, for a while, negative. This is contrary to our hypothesis.

2. L is never positive when all roots are real (Art. 236).

3. $2^{11}L - J^3$ cannot change its sign while J is negative and $D > 0$. This may be proved by showing that

$$2^{11}L = J^3 \tag{1}$$

and

$$2^7 K = J^2 - J^2 x \tag{2}$$

(where x is a positive quantity) are inconsistent while J remains negative.

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* The references are to Salmon's *Modern Higher Algebra*, fourth edition.

Let us substitute the above values of K and L in the second member of the identity,

$$16I^2 = JK^4 + 8LK^3 - 2J^2LK^2 - 72JKL^2 - 432L^3 + J^3L^2 \geqq 0. \qquad (3)$$

This gives, after dividing by the negative factor $J^9/2^{29}$ and arranging the result with reference to x,

$$2x^4 - 9x^3 - 17x^2 + 125x + z = 0, \qquad (4)$$

(where z is a quantity never negative). If we can show that the equation just written has no real positive roots, we prove the equations $2^{11}L = J^3$ and $2^7K = J^2 - J^2x$ to be inconsistent. For this quartic the quantities*

$$b^2 - ac, = \tfrac{515}{48},$$

and

$$3aT + 2(b^2 - ac)S, = -\tfrac{515}{24}z - \tfrac{190059}{72}, \qquad (5)$$

are always positive and negative respectively. Hence, if the discriminant of the quartic be positive, all four values of x are imaginary. We next suppose this discriminant to be negative; then the quartic has, of course, two real roots. Both of these roots are positive or both negative; for if we let x equal zero in the quartic the result is positive; and, again, if we let $x = \infty$ or $-\infty$, either result is positive. Hence, on either side of zero there are either two or no real roots. With a special value of z we find that the two roots are negative, and being once negative in a special case they will be so in general, because $z/2$ is the product of all the roots, and when z varies from the special value no root can pass through zero (and so change its sign) unless z does; but z, by hypothesis, is never negative.† Hence, $2^{11}L = J^3$ and $2^7K = J^2 - J^2x$ are inconsistent while J is negative.‡

4. Let us first suppose all roots of the quintic to be real, and distinct (since D is not equal to zero). The sign of $2^{11}L - J^3$ is found to be positive for a special quintic whose roots are all real. It must be positive for *every* quintic whose roots are all real and distinct. This will be established if we

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* Art. 206, note.

† When $z = 0$, one of the two real roots is, of course, zero.

‡ When the discriminant of the quartic is zero, z is about 215, and the two real roots about -2.1 each.

show that $2^{11}L - J^3$ can then have no zeros. $2^{11}L - J^3$ being positive for a special case infers J negative for the same case (since L is not positive). But unless J *has* changed sign or become zero, $2^{11}L - J^3$ cannot become zero, as has just been proved in § 3. J cannot pass zero simultaneously with $2^{11}L - J^3$, because L is negative and not zero when J is zero in the quintics under consideration.* Therefore, J must always be negative when all roots are real and distinct, and, consequently, $2^{11}L - J^3$ must then be positive. Hence the first criterion: *Whenever the roots are all real and distinct, both* L *and* $J^3 - 2^{11}L$ *are negative.*

5. Let us next suppose four roots to be imaginary; J may be either negative or positive. *J negative.* Now, whichever sign $2^{11}L - J^3$ has in a special case, it will have the same sign in the general case under consideration, as was proved in § 3. The sign as thus determined would be minus (as might have been inferred from the fact that $2^{11}L - J^3$ passes through zero when the line passes through the double tangent). We conclude, therefore, that whenever four roots are imaginary and J negative, $2^{11}L - J^3$ is also negative (or zero). *J positive or zero.* When all roots are real and distinct J is always negative (§ 4); therefore, if it is positive or zero four roots must be imaginary. Hence the second criterion: *Whenever four of the roots are imaginary, at least one of the quantities* J, $J^3 - 2^{11}L$ *is not negative.*

III. THE MORE GENERAL CRITERION $2^{11}L - J^3 + \mu JD$.

6. Sylvester has shown that this criterion may be substituted for $2^{11}L - J^3$, provided μ be so taken that

$$-2 \gtreqless \mu \gtreqless 1.$$

The following investigation establishes his result:—

In order that a criterion behave like $2^{11}L - J^3$ in reference to the reality of the roots,

(a) It must not change sign while D is positive (not zero) and J is negative;

(b) It must change sign as we pass through two pairs of equal roots (J remaining, of course, negative, and D becoming zero for the instant).

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* Suppose $L = J = 0$. By linear substitution the quintic can be put into the form $ax^5 + 5exy^4$, which when $D > 0$ has four imaginary roots, proportional to the fourth roots of -1, and one real root equal to zero. No linear substitution can transform this quintic into one having five real roots.

It is obvious that $2^{11}L - J^3 + \mu JD$ satisfies the latter requirement. For, we have seen (§ 5) that $2^{11}L - J^3$ then changes sign,—it passes *through* zero. D, on the other hand, does not change sign, but decreases to zero, remains stationary for an instant, and then again assumes positive values,—it *touches* zero. Hence, the sign of $2^{11}L - J^3 + \mu JD$ depends upon that of $2^{11}L - J^3$ in the vicinity of zero, the values of D then being very small in comparison with those of $2^{11}L - J^3$.

To prove that when J is negative and D positive (not zero), $2^{11}L - J^3 + \mu JD$ cannot change sign while μ lies within certain limits.

As in § 3, so here, we are to prove that certain equations

$$2^{11}L = J^3 - \mu JD \tag{6}$$

and

$$2^7 K = J^2 - J^2 x \tag{7}$$

(where x is a positive quantity) are inconsistent while J is negative. From these equations,

$$L = \frac{J(J^2 - \mu x)}{2^{11}}, \qquad K = \frac{J^2 - J^2 x}{2^7}. \qquad (8)\ (9)$$

Substituting these values in the expression for $16I^2$ (§ 3), we have, after dividing out $J^9/2^{29}$,

$$\begin{aligned}(\mu + 2)x^4 &+ (27\mu^3 + 72\mu^2 + 29\mu - 9)x^3 \\ &- (25\mu^2 + 205\mu + 17)x^2 + 125x + z = 0 \end{aligned} \tag{10}$$

(where z is a quantity never negative; viz. $- 2^{33}I^2/J^9$).

Since $x (= D/J^2)$ must always be positive, the problem is to find what values of μ will cause the equation just written to have no real positive roots.

When $z = 0$ one of the roots of equation (10) is zero; consequently that equation will have two or four real roots according as its discriminant is negative or positive. Two adjacent roots of this discriminant are $\mu = 1$, $\mu = -\frac{83}{45}$; between these values it is negative, and immediately beyond, positive. No root of equation (10), where $z = 0$, can change its sign if $\mu > -2$, because $\frac{125}{\mu + 2}$, the product of the three remaining roots with sign changed, is always positive (cf. § 3). Any special case shows that if $-\frac{83}{45} < \mu < 1$ neither of the real roots are positive; and if $-2 < \mu < -\frac{83}{45}$ not one of the four real roots is positive. But they cannot change sign while $\mu > -2$ and $z = 0$.

When $z = 0$ the real roots of equation (10) cannot have opposite signs. If z be increased at pleasure the real roots will in no case take opposite signs. This follows from the fact that z is a continuous and one-valued function of x. [Equation (10) may be written $z' + F(x) = 0$; its roots are given by the intersection of $z = z'$ and $z = -F(x)$.]

This shows that equation (10) has no real positive roots when $-2 < \mu < 1$.

The same thing may be readily seen upon the introduction of certain geometrical considerations:—

7. Let us regard equation (10) as the equation of a system of curves whose abscissas are x, whose ordinates are μ, while z is the parameter. Arranged with reference to μ, equation (10) becomes

$$(27x^3)\mu^3 + (72x^3 - 25x^2)\mu^2 + (x^4 + 29x^3 - 205x^2)\mu$$
$$+ (2x^4 - 9x^3 - 17x^2 + 125x + z) = 0. \qquad (11)$$

To each value of x, in any particular curve, correspond three, and but three, values of μ. From equations (10) and (11) we see that the lines $\mu = -2$ and $x = 0$ are asymptotes to the system. None of these curves can cross the x-axis on the positive side of the μ-axis; for, if we make $\mu = 0$ in equation (11) we have

$$2x^4 - 9x^3 - 17x^2 + 125x + z = 0,$$

which, according to § 3, has no real positive roots.

8. If for a certain value of z we draw the corresponding curve; and if for a certain other value of z greater than the former, we likewise draw another curve: then to a common abscissa corresponds, in general, a set of ordinates to each curve; the absolute values of the ordinates belonging to the second curve are severally greater than the absolute values of those belonging to the first curve, x and z being positive.

Let α, β, γ be the roots of equation (11) for a particular x and z. Let $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$ be the roots of equation (11) for the same x but when z becomes $z + dz$ (dz, like z and x, is positive). We have

$$\begin{aligned} \alpha + \beta + \gamma &= \text{const.}, \\ \alpha\beta + \beta\gamma + \gamma\alpha &= \text{const.}, \\ \alpha\beta\gamma &= -z' + \text{const.} \end{aligned}$$

(where $z' = z/27x^3$). These differentiated give

$$\begin{aligned} d\alpha + d\beta + d\gamma &= 0, \\ (\beta + \gamma)\,d\alpha + (\gamma + \alpha)\,d\beta + (\alpha + \beta)\,d\gamma &= 0, \\ \beta\gamma\,d\alpha + \gamma\alpha\,d\beta + \alpha\beta\,d\gamma &= -dz'; \end{aligned}$$

$$\therefore\ d\alpha = -\frac{dz'}{\varDelta}\begin{vmatrix} 1 & 1 \\ \gamma + \alpha & \alpha + \beta \end{vmatrix}, \tag{12}$$

$$d\beta = -\frac{dz'}{\varDelta}\begin{vmatrix} 1 & 1 \\ \beta + \gamma & \alpha + \beta \end{vmatrix}, \tag{13}$$

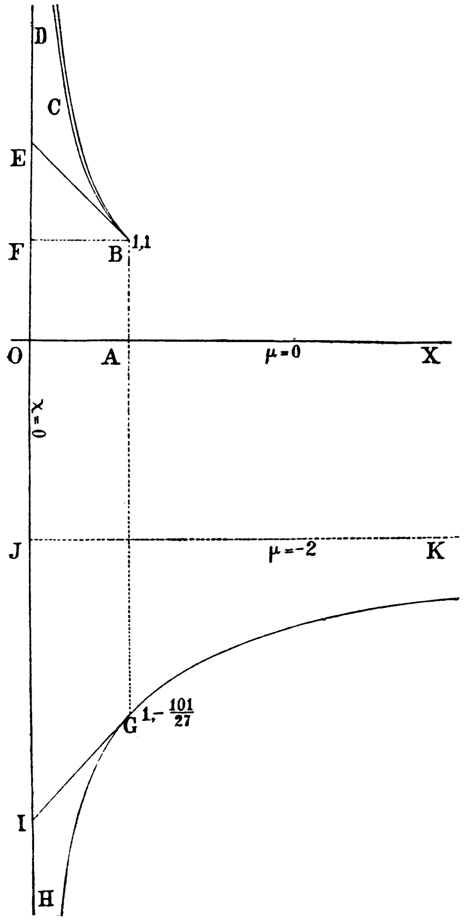
$$d\gamma = -\frac{dz'}{\varDelta}\begin{vmatrix} 1 & 1 \\ \beta + \gamma & \gamma + \alpha \end{vmatrix}, \tag{14}$$

where

$$\varDelta = \begin{vmatrix} 1 & 1 & 1 \\ \beta + \gamma & \gamma + \alpha & \alpha + \beta \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\alpha - \beta)(\beta - \gamma)(\alpha - \gamma). \tag{15}$$

If $\varDelta_1$, $\varDelta_2$, $\varDelta_3$ denote the minors written above, we have

$$\varDelta_1 = \beta - \gamma, \quad \varDelta_2 = \alpha - \gamma, \quad \varDelta_3 = \alpha - \beta.$$



From these values of the determinants, it is easily seen that

If $\beta > \alpha > 0 > \gamma$ be any three real quantities, then Δ_1/Δ, Δ_2/Δ are always negative; and Δ_3/Δ is always positive whether α, β are real or conjugate imaginaries.

Thus we see that the increment of each of the positive quantities is positive, while that of the negative quantity is negative. Hence, if we continue to increase z the ordinates will continue to increase in absolute value, provided we can show that for $z = 0$ two of the roots of equation (11) are always either real and positive or conjugate imaginaries, while one is always real and negative. For a special value of x (z being zero) we find this to be the case; it will be so for all other positive values of x, because, as we saw in § 7, no branch of any curve can cross the x-axis on the positive side of the μ-axis.

9. Let us draw the curve whose z is equal to zero, drawing only that part which lies upon the positive side of the μ-axis (as in the figure). The curve breaks up into

$$x = 0$$

and

$$(27x^2)\mu^3 + (72x^2 - 25x)\mu^2 + (x^3 + 29x^2 - 205x)\mu + (2x^3 - 9x^2 - 17x + 125) = 0.^* \qquad (16)$$

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* If for x we write y, and for μ, $-\frac{x}{y}$, equation (16) may be written

$$\varphi(x, y) = 0.$$

This is the equation of Sylvester's "Bicorn." Cayley, in his Eighth Memoir on Quantics (*Phil. Trans.*, 1867, p. 527), and Salmon, in his Modern Higher Algebra (§ 243, fourth edition), have written through mistake $-3x^3$ and $-x^2$ for the terms $-27x^3$ and $-25x^2$ in the expression $\varphi(x, y)$.

As x passes through unity the sign of the discriminant of this cubic changes from — to + ; i. e. μ has three real values before the passage and but one real value after it.

We find that the curve (16) has a ramphoid cusp at the point (1,1), and that the equation of the tangent at that point is

$$x + \mu = 2. \tag{17}$$

10. From what has just been proved (§ 8), we see that any curve whose z is positive lies beyond the one whose z is zero, always measuring from the x-axis and confining ourselves, of course, to the positive side of the μ-axis. In other words, the regions $OABCDO$ and $OXKGHO$ will not be intrenched upon by any curve having a positive z. Consequently, no ordinate whose value lies between — 2 and + 1, both inclusive, can reach any curve having a positive parameter z. That is, whenever μ lies between the limits — 2 and + 1, equation (10) contains no real positive roots. Hence, $2^{11}L - J^3 + \mu JD$ is equivalent, as a criterion, to $2^{11}L - J^3$.*

IV. OTHER SETS OF CRITERIA.

11. *To prove that K is always positive when the roots of the quintic are all real and distinct.* This may be done by showing that K cannot change sign while J is negative, and D and $2^{11}L - J^3 + \mu JD$ are positive.

Let us suppose K to become zero, as it must if it change sign ; then the expression for $16I^2$, in § 3, becomes

$$- 432L + J^3 \underset{>}{=} 0. \tag{18}$$

This is certainly absurd when L is positive and J is negative. We have, then, to consider the possibility of this relation, only when L is negative and J is negative.

By hypothesis

$$2^{11}L - J^3 + \mu JD > 0.$$

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* When $z = 0$ and $\mu = 1$, there is an apparent exception ; for then $x = 1$. But this infers $D = J^2$, which is contrary to assumptions made in § 6 (b). So when $z = 0$ and $\mu = -2$; then $x = \infty$. This infers $J = 0$, which is contrary to an assumption made in § 6 (a).

Adding to this $\frac{128}{27}$ times relation (18), we obtain

$$\tfrac{101}{27}J^3 + \mu JD \geqq 0,$$

or

$$\tfrac{101}{27} + \mu \frac{D}{J^2} \leqq 0.$$

This is certainly absurd if

$$\mu > -\tfrac{101}{27} \div \frac{D}{J^2}.$$

It is, then, absurd if $\mu = 0$; i. e. when $2^{11}L - J^3 > 0$. But whenever $2^{11}L - J^3 > 0$ (J and D conditioned as above), so is $2^{11}L - J^3 + \mu JD > 0$ (μ having any value which will cause equation (10) to have no real positive roots), and conversely. Hence, for any of these values of μ an absurdity results. The consequence is that K cannot become zero, and so change sign, under the given hypotheses. By taking a special case we see that K is positive.

12. *Extended range of the values of μ for particular quintics.* We have done more than to establish common limits for μ. Given a particular quintic we form D/J^2 $(= x)$; the corresponding ordinate, taken from the figure, shows how μ may be chosen. Had we a chart of the system of curves represented by equation (11),—drawn once for all,—we could, by calculating D/J^2 and $-16I^2/J^9$ (i. e. x and z), obtain a still wider range of values for μ for any particular quintic.

REMARK. We can, if we choose, regard equation (11) as that of a surface, where z is perpendicular to the plane of the figure. The horizontal projections of the contour lines of this surface constitute the system just mentioned. The surface continually recedes from the xz-plane as z increases (§ 8).

Given x $(= D/J^2)$, we have, in general, to solve the cubic equation (11) in order to obtain the limiting values of μ. It may be worth while to note a slight extension of the limits which does not require this labor.

(a) The upper limit. If x, μ be co-ordinates of any point situated above the x-axis and below the curve, they will cause equation (10) to have no real positive roots; surely the co-ordinates of any point above the x-axis and below the tangent at (1,1) will do likewise. The equation of the tangent at (1,1) is $x + \mu = 2$; i. e. $\mu = 1 + 128K/J^2$, or $1 + r$, where $r = 128K/J^2$. Hence, for a given quintic whose K is positive, μ can always have an upper limit as great as $1 + r$.

REMARK. $2^{11}L - J^3 + (1 + r)JD = 2^{11}L - (128K)^2/J$; consequently when K is positive

$$8K^2 - JL$$

is equivalent, as a criterion, to $2^{11}L - J^3$, etc.

(b) The lower limit. The co-ordinates of the point which lies upon the curve directly below the point (1,1) are 1, $-\frac{101}{27}$. The direction of the tangent at this point is given by $\tan^{-1}\frac{29}{27}$. Hence, for a given quintic whose K is positive, μ can have any value whatever between

$$-\tfrac{101}{27} - \tfrac{29}{27}r \quad \text{and} \quad 1 + r.$$

13. The figure enables us to obtain an infinitude of expressions any one of which is equivalent, as a criterion, to $2^{11}L - J^3$.

For instance, we see that μ may be any function of x, provided the values of the function are positive when x is positive, and do not extend to the curve when x is less than unity. In particular,

$$\mu = p^{-K},$$

where p is any real positive quantity greater than unity, is such a function of x. Consequently

$$2^{11}L - J^3 + p^{-K}JD$$

is equivalent, as a criterion, to $2^{11}L - J^3$.

14. *Hermite's first set of criteria.* Sylvester's results taken in connection with § 11 and § 12(a) enable us to verify, without much difficulty, Hermite's first set of conditions. But as they are not suggested by the foregoing discussion, the proof will be omitted. By way of comparison, three sets of necessary and sufficient conditions for five real and distinct roots are given here:

$D=+$,	$2^{11}L-J^3+\mu JD=+$,	$J=-$,		Sylvester's.
$D=+$,	$8K^2-JL=+$,	$J=-$,	$K=+$,	§ 12(a).
$D=+$,	$2^{11}L-J^3+JD=+$,	$K(JL+K^2)-18L^2=+$,	$K=+$,	Hermite's.